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LETTER TO THE EDITOR

Two-step renormalisation group approach for randomly diluted Ising models†

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Abstract. A two-step renormalisation group approach—a Migdal-Kadanoff (or decimation) transformation (MK) followed by a mean-field renormalisation group (MFRG)—has been applied to random bond Ising models on d-dimensional hypercubic lattices. Calculations of the critical couplings and percolation concentrations for two and three dimensions show much improvement, even for small size clusters, compared with the results of separate MK or MFRG calculations. Critical exponents ν_T and ν_p are also improved in some cases. Estimates of critical curves in the concentration-temperature plane are also obtained for the two- and three-dimensional models.

Various approximate real-space renormalisation group (RG) methods have been developed to study the critical behaviour of pure and random spin systems (see Burkhardt and van Leeuven (1982) for reviews and detailed references). In some cases these methods give approximate results for the critical properties of the systems, but in general they are poor compared with known exact results. Some of the very commonly used methods for the Ising models are the Migdal-Kadanoff (MK) transformations (Migdal 1976, Kadanoff 1976), the decimation methods (Barber 1975, Nelson and Fisher 1975, Kadanoff and Houghton 1975, Young and Stinchcombe 1976) and the recently proposed mean-field renormalisation group (MFRG) method (Indekeu *et al* 1982). In all these cases the results can be improved to some extent by considering large-size cells, which involve lengthy calculations and slow convergence.

The MK or decimation methods consider only the interactions among the spins of a finite cluster, neglecting the effect of the surrounding spins and thus underestimating the interactions among the spins, which leads to a lower value of critical temperature. When applied to diluted systems these methods lead to a higher value of the percolation or critical concentration as a consequence of the underestimation of the interactions. On the other hand, in MFRG the interactions within the cluster are treated exactly and the effect of the neighbouring spins is taken into account in terms of a mean field acting on each spin of the cluster. This effectively leads to an overestimation of the interactions among the spins, giving a higher value of the critical temperature and a lower value for the critical concentration for the diluted system.

Recently de Alcantara Bonfim *et al* (1984) have presented a two-step renormalisation group method, combining the MFRG and decimation methods, and have applied it to pure Ising models on various lattices. This method compensates for the contrary effects

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of the two methods used, thus leading to substantial improvements both in critical couplings and exponents. However, application of the method of de Alcantara Bonfim *et al* is limited to systems with only one parameter space, e.g. pure Ising models, etc. As the first step of this method, i.e. MFRG, leads to a single approximate recursion relation (Indekeu *et al* 1982, Droz *et al* 1982, Plascak 1984a, b) among the various parameters of the Hamiltonian, it is not possible to apply it directly to systems with two or more parameters. This is because separately defined parameters (i.e. a number of recursion relations equal to the number of parameters of the Hamiltonian) are needed in order to apply, in the second step, a decimation or MK transformation to the renormalised lattice obtained through the application of MFRG in the first step. However, if the order of application of the two steps is reversed, a decimation or MK RG followed by MFRG, it would be possible to realise a two-step renormalisation group procedure for systems with two or more parameters, such as a randomly diluted Ising model.

In this letter we apply a two-step RG method to random bond-diluted Ising models on d-dimensional hypercubic lattices. In this method we first apply a MK (or decimation in two dimensions) method, obtaining two separate recursion relations for the renormalised couplings and renormalised concentration of bonds (Jayaprakash *et al* 1978, Yeomans and Stinchcombe 1979). This renormalised lattice is then treated within the MFRG, from which a single approximate recursion relation is obtained for the final renormalised parameters in terms of the original lattice parameters. This final recursion relation is used to study the critical properties of the system.

In order to illustrate our approach we consider a bond-diluted Ising model on a d-dimensional lattice, with an effective Hamiltonian

$$\mathcal{H} = \sum_{ij} K_{ij} S_i S_j \tag{1}$$

where $S_i = \pm 1$ are the Ising spins. The nearest-neighbour couplings, defined by $K_{ij} = J_{ij}/kT$, are random variables, having a probability distribution

$$P(K_{ij}) = p\delta(K_{ij} - K) + (1-p)\delta(K_{ij}) \tag{2}$$

where p is the concentration of bonds and K is the coupling of the pure system. In the first step of our approach we apply the MK transformation (Jayaprakash *et al* 1978) to the cells of linear size b_1, $(b_1+1)^d$ spins (we shall be taking $b_1 = 2$). This leads to the following recursion relations for the renormalised concentration p' and coupling K':

$$p' = 1 - (1-p^2)^{\tilde{D}} \tag{3}$$

$$p'K' = \tfrac{1}{2}p^2\tilde{D}\ln(\cosh 2K) \tag{4}$$

where

$$\tilde{D} = b_1^{(d-1)}.$$

In the second step we apply the MFRG to the renormalised lattice defined by the parameters (p', K'). In this method (Indekeu *et al* 1982) one compares the two clusters of different sizes, with a scaling factor of $b_2 = (N/N')^{1/d}$, N and N' being the number of spins in the two clusters and $N > N'$. The interactions of the neighbouring spins are simulated by a mean field $\bar{S}$ acting on each spin of the cluster and the RG transformation is realised by considering the magnetisations of the clusters to scale in the same way as their respective surrounding mean fields. Realising this transformation for the case of one-spin ($N' = 1$) and two-spin ($N = 2$) clusters gives the following

single recursion relation for the renormalised parameters (p'', K'') in terms of (p', K'):

$$2dp''K'' = p'(1-p')(2d-1)K' + 2(p')^2(2d-1)K'\frac{\exp(2K')}{\exp(2K')+1}. \tag{5}$$

Substituting p' and K' from equations (3) and (4) gives the following final single approximate recursion relation for the renormalised parameters (p'', K'') in terms of the original parameters (p, K):

$$2dp''K'' = \tfrac{1}{2}\tilde{D}p^2(2d-1)(1-p^2)^{\tilde{D}}\ln(\cosh 2K) + \tilde{D}p^2(2d-1)[1-(1-p^2)^{\tilde{D}}]$$
$$\times \ln(\cosh 2K)\frac{(\cosh 2K)^q}{(\cosh 2K)^q+1} \tag{6}$$

where

$$q = \tilde{D}p^2/[1-(1-p^2)^{\tilde{D}}]. \tag{7}$$

The effective final scaling parameter for the present method is $b = b_1 b_2$. It is easy to see from equation (6) that at the pure Ising fixed point $(p = 1)$ the criitcal coupling is defined by

$$K'' = \tilde{D}\frac{(2d-1)}{2d}\ln(\cosh 2K)\frac{(\cosh 2K)^{\tilde{D}}}{(\cosh 2K)^{\tilde{D}}+1}. \tag{8}$$

On the other hand, at the percolation fixed point $(K = \infty)$ the critical concentration is given by

$$p'' = \frac{(2d-1)}{2d}\tilde{D}p^2[2-(1-p^2)^{\tilde{D}}]. \tag{9}$$

Linearising the recursion relation, equation (6), around the fixed points gives the critical exponents ν_T and ν_p through the relation

$$\left.\frac{\partial\mu''}{\partial\mu}\right|_{\mu^*} = b^{1/\nu_\mu} \tag{10}$$

where μ can be K or p.

We used equations (6)-(9) to calculate the critical couplings, critical concentrations and exponents for 2D and 3D bond-diluted Ising models. Besides the one-spin and two-spin clusters $(b_2 = (2)^{1/d})$ in the second step (MFRG) of our approach, we also considered the four-spin clusters $(b_2 = (4)^{1/d})$. The results of our calculations for 2D and 3D lattices for various scaling parameters b are presented in tables 1 and 2 respectively. It is clear from the tables that our approach gives better values for the critical couplings K_c and critical concentrations p_c, closer to the known exact or series results, compared with separate MK or MFRG methods. It is worth mentioning that, without considering large-size cells, in the present method it is possible to obtain very good estimates for the critical properties of the system under consideration. The values of the critical exponents are also improved in some cases, particularly in two dimensions. The best estimates for the critical couplings and critical concentrations are obtained, as also found by de Acantara Bonfim *et al*, for the most symmetric clusters considered in the MFGR part of the two-step RG method.

Although in the present RG method it is not possible to determine the complete renormalisation flow in the parameter space of the Hamiltonian, equation (1), because of the single recursion relation, equation (6), one can estimate the phase boundary in

Table 1. Critical points and exponents for the 2D bond-diluted Ising model on a square lattice.

		Pure Ising fixed-point		Percolation fixed-point	
Method	b	K_c	ν_T	P_c	ν_p
MFRG	$\sqrt{2}$	0.346	1.667	0.333	1.553
	$\sqrt{4}$	0.360	1.440	0.354	1.398
MK (decimation)	$\sqrt{2}$	0.609	0.669	0.618	0.818
	2	0.609	1.337	0.618	1.635
Two-step RG (our method)	$\sqrt{4}$	0.536	0.883	0.475	0.757
	$\sqrt{8}$	0.536	1.325	0.475	1.136
	$\sqrt{8}$	0.506	0.966	0.468	0.893
	$\sqrt{16}$	0.506	1.288	0.468	1.190
Exact results		0.441[a]	1	0.5[b]	1.34[c]

[a] Onsager (1944).
[b] Sykes and Essam (1964).
[c] Wallace and Young (1978).

Table 2. Critical points and exponents for the 3D bond-diluted Ising model on a simple cubic lattice.

		Pure Ising fixed-point		Percolation fixed-point	
Method	b	K_c	ν_T	P_c	ν_p
MFRG	$2^{1/3}$	0.203	1.025	0.2	1.0
	$4^{1/3}$	0.205	0.925	0.203	1.372
MK	2	0.261	1.068	0.28	1.23
Two-step RG (our method)	$(16)^{1/3}$	0.252	1.12	0.246	1.086
	$(32)^{1/3}$	0.246	1.13	0.243	1.13
Exact (series) results		0.221[a]	0.64[b]	0.25[c]	0.82[d]

[a] Pawley *et al* (1983).
[b] Fisher and Burford (1967).
[c] Sykes *et al* (1976).
[d] Dunn *et al* (1975).

the concentration (p)-temperature ($T = K^{-1}$) plane using the fixed point relation, i.e. equation (6) with $p'' = p$ and $K'' = K$ (for a discussion see Droz *et al* 1982, Plascak 1984a, b). Figure 1 shows our results for the critical curves in the pK^{-1} plane for the 2D and 3D models, with a scaling factor $b = \sqrt{8}$. One can also estimate the limiting slope

$$\frac{1}{T_c(1)}\left(\frac{d}{dp}T_c(p)\right)_{p=1}$$

of the critical curve at the Ising fixed point. For the 2D lattice, with a scaling factor $b = \sqrt{8}$, we get a value of 1.3 which is close to the exact result of Harris (1974), 1.329, compared with the value obtained by the decimation method 1.377 (Yeomans and Stinchcombe 1979).

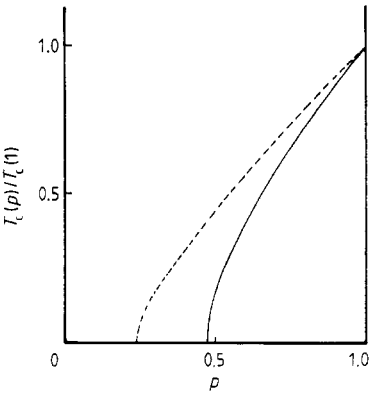

Figure 1. Critical curves for the 2D (——, $b=\sqrt{8}$) and 3D (---, $b=(16)^{1/3}$) lattices.

It should be noted that the basic principle of the MFRG (Indekeu *et al* 1982), and consequently of the present TSRG, is very similar to that of the well known phenomenological finite-size scaling method (Nightingale 1976, 1977), in which a single renormalisation recursion relation is obtained through the scaling of correlation length and is used to estimate the phase diagram and critical properties. Furthermore, the present RG method is free from the proliferation problem, as is also the case with the finite-size scaling method, contrary to the conventional microscopic RG methods.

In conclusion we have presented a two-step renormalisation group approach and applied it to the random bond Ising model, considerably improving the results for critical properties, even for small-size clusters, compared with other frequently used real space renormalisation group methods. It should be noted that our method is very general and can be applied to any other system with two or more parameters in the Hamiltonian. We are applying this method to some other systems of interest and hope to publish the results in the near future.

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